PRIIMEK	IME	VPISNA ŠTEVILKA	SMER

NALOGA	TOČKE
1.	
2.	
3.	
4.	
SKUPAJ	

RAČUNSKI DEL IZPITA IZ MATEMATIČNE ANALIZE II (16.6.2008)

Navodilo: vsako nalogo rešuj na strani, kjer je napisana. Če bo naloga reševana kje drugje, mora biti to posebej označeno. Veliko uspeha pri reševanju!

1. naloga: Izračunaj dolžino krivulje

$$\text{(a.)}\ y = \ln(1-x^2),\ x \in [0, \tfrac{1}{2}], \qquad \text{(b.)}\ y = \sqrt{x-x^2} + \arcsin\sqrt{x},\ x \in [0,1]\ .$$

Pomoč: Dolžino krivulje $y = y(x)$ na intervalu $[a,b]$ se izračuna $\int_a^b \sqrt{1+y'^2}\,dx$, v pomoč vam bo mogoče tudi $\frac{1}{1-x^2} = \frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right)$.

(30 točk)

a) $y' = \frac{1}{1-x^2}\cdot(-2x) = -\frac{2x}{1-x^2}, \quad y'^2 = \frac{4x^2}{(1-x^2)^2}, \quad 1+y'^2 = \frac{(1-x^2)^2+4x^2}{(1-x^2)^2} = \frac{1-2x^2+x^4+4x^2}{(1-x^2)^2} =$

$= \frac{1+2x^2+x^4}{(1-x^2)^2} = \frac{(1+x^2)^2}{(1-x^2)^2}$

$S_1 = \int_0^{\frac{1}{2}} \sqrt{\frac{(1+x^2)^2}{(1-x^2)^2}}\,dx = \int_0^{\frac{1}{2}} \frac{1+x^2-1+1}{1-x^2}\,dx = \int_0^{\frac{1}{2}} \frac{2}{1-x^2}\,dx - \int_0^{\frac{1}{2}} dx = \cancel{2}\frac{1}{\cancel{2}}\left(\int_0^{\frac{1}{2}} \frac{1}{1+x}dx + \int_0^{\frac{1}{2}} \frac{1}{1-x}dx\right) - x\Big|_0^{\frac{1}{2}} =$

$= \ln|1+x|\Big|_0^{\frac{1}{2}} - \ln|1-x|\Big|_0^{\frac{1}{2}} - \frac{1}{2} = \ln\frac{3}{2} - \ln\frac{1}{2} - \frac{1}{2} = \underline{\underline{\ln 3 - \frac{1}{2}}}$

b.) $y' = \frac{1}{2\sqrt{x-x^2}}\cdot(1-2x) + \underbrace{\frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}}}_{2\sqrt{x-x^2}} = \frac{1-2x+1}{2\sqrt{x-x^2}} = \frac{1-x}{\sqrt{x-x^2}}$

$1+y'^2 = \frac{x-x^2+(1-x)^2}{x-x^2} = \frac{x-\cancel{x^2}+1-2x+\cancel{x^2}}{x-x^2} = \frac{\cancel{1-x}}{x\cdot\cancel{(1-x)}} = \frac{1}{x}$

$S_2 = \int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\downarrow 0}\int_a^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\downarrow 0} \frac{\sqrt{x}}{\frac{1}{2}}\Big|_a^1 = 2\lim_{a\downarrow 0}(1-\sqrt{a}) = \underline{\underline{2}}$

2. naloga: Zapiši Taylorjevo formulo reda 5 za $f(x) = \cos x$ in z njeno pomočjo izračunaj približno vrednost $\int_0^1 \frac{1-\cos x^2}{x^3}dx$. Oceni napako, ki jo pri tem narediš. (25 točk)

$f(x) = \cos x \rightsquigarrow 1$

$f'(x) = -\sin x \rightsquigarrow 0$

$f''(x) = -\cos x \rightsquigarrow -1$

$f'''(x) = \sin x \rightsquigarrow 0$

$f^{IV}(x) = \cos x \rightsquigarrow 1$

$f^{V}(x) = -\sin x \rightsquigarrow 0$

$f^{VI}(x) = -\cos x$

$$f(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + r_5$$

$$r_5 = -\frac{\cos \Theta x}{6!}x^6, \quad \Theta \in [0,1]$$

$$\cos x^2 = f(x^2) = 1 - \frac{x^4}{2} + \frac{x^8}{4!} + r_5^*$$

$$r_5^* = -\frac{\cos \Theta x^2}{6!}x^{12}$$

napaka = R

$$\int_0^1 \frac{1-\cos x^2}{x^3}dx = \int_0^1 \frac{\cancel{1} - \cancel{1} + \frac{x^4}{2} - \frac{x^8}{4!} - r_5^*}{x^3}dx = \int_0^1 \left(\frac{x}{2} - \frac{x^5}{4!}\right)dx - \int_0^1 \frac{r_5^*}{x^3}dx \approx$$

$$\approx \left(\frac{x^2}{4} - \frac{x^6}{6 \cdot 4!}\right)\Bigg|_0^1 = \frac{1}{4} - \frac{1}{6 \cdot 4!}$$

Ocena napake

$$0 < R = -\int_0^1 -\frac{\cos \Theta x^2}{6!\, x^3}x^{12}dx = \frac{1}{6!}\int_0^1 \underbrace{\cos \Theta x^2}_{0 \leq \cdot \leq 1} \cdot x^9 dx \leq \frac{1}{6!}\int_0^1 x^9 dx =$$

$$= \frac{1}{6!}\frac{x^{10}}{10}\Bigg|_0^1 = \frac{1}{10 \cdot 6!} \rightarrow 0 < R < \frac{1}{10 \cdot 6!}$$

3. naloga: S pomočjo verižnega pravila izračunaj Jacobijevo matriko za preslikavo $G \circ F$, kjer sta $G(x,y) = (e^{x+2y}, \sin(y-2x))$ in $F(u,v,w) = (u + 2v^2 + 3w^3, 2u - v^2)$. (20 točk)

$$J_G(x,y) = \begin{bmatrix} e^{x+2y} & 2e^{x+2y} \\ -2\cos(y-2x) & \cos(y-2x) \end{bmatrix}$$

$$J_G(F(u,v,w)) = J_G(u+2v^2+3w^3, 2u-v^2) = \begin{bmatrix} e^{\overbrace{u+2v^2+3w^3+4u-2v^2}^{5u+3w^2}} & 2e^{5u+3w^2} \\ -2\cos(\underbrace{2u-v^2-2u-4v^2-6w^3}_{-5v^2-6w^3}) & \cos(5v^2+6w^3) \end{bmatrix}$$

$$J_F(u,v,w) = \begin{bmatrix} 1 & 4v & 9w^2 \\ 2 & -2v & 0 \end{bmatrix}$$

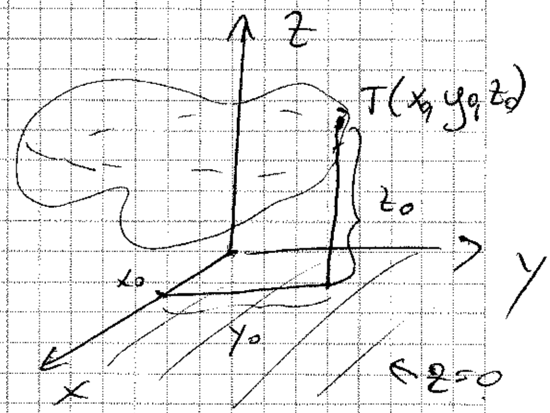
$$J_{G\circ F}(u,v,w) = \begin{bmatrix} e^{5u+3w^2} & 2e^{5u+3w^2} \\ -2\cos(5v^2+6w^3) & \cos(5v^2+6w^3) \end{bmatrix} \cdot \begin{bmatrix} 1 & 4v & 9w^2 \\ 2 & -2v & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 5e^{5u+3w^2} & 0 & 9w^2\, e^{5u+3w^2} \\ 0 & -10v\cos(5v^2+6w^3) & -18w^2\cos(5v^2+6w^3) \end{bmatrix}$$

4. naloga: Katere točke na elipsoidu $2x^2 + 3y^2 + 2z^2 + 2xz = 6$ so najbolj oddaljene od ravnine $z = 0$? (25 točk)

Odd. $\overset{(x_0, y_0, z_0)}{T}$ od ravnine $z=0$ $= z_0$

$$d(T, z=0) = z \quad \left(= d((x,y,z), z=0)\right)$$



$$F(x,y,z,\lambda) = z + \lambda \cdot (2x^2 + 3y^2 + 2z^2 + 2xz - 6)$$

$$\frac{\partial F}{\partial x} = 4\lambda x + 2\lambda z = 0$$

$$\frac{\partial F}{\partial y} = 6\lambda y = 0 \iff \overset{\text{I}}{\lambda = 0} \text{ ali } \overset{\text{II}}{y = 0}$$

$$\frac{\partial F}{\partial z} = 1 + \lambda(4z + 2x) = 0$$

I $\lambda = 0 \Rightarrow 1 + 0 \cdot (2z + 2x) = 0$ ↯ nikoli

$$\frac{\partial F}{\partial \lambda} = 2x^2 + 3y^2 + 2z^2 + 2xz - 6 = 0$$

II $y = 0$

$$2x^2 + 2z^2 + 2xz - 6 = 0$$

$$\lambda(4x + 2z) = 0 \overset{\lambda \neq 0}{\Longrightarrow} z = -2x$$

$$1 + \lambda(4z + 2x) = 0$$

$$2x^2 + 2 \cdot 4x^2 - 4x^2 - 6 = 0$$

$$6x^2 = 6$$

$$x = \pm 1 \Rightarrow z = \mp 2$$

$$\left.\begin{array}{l} T_1(1, 0, -2) \\ T_2(-1, 0, 2) \end{array}\right\} d(T, z=0) = 2$$

↑ maksimalno oddaljeni točki