PRIIMEK	IME	VPISNA ŠTEVILKA	SMER

NALOGA	TOČKE
1.	
2.	
3.	
4.	
SKUPAJ	

MATEMATIČNA ANALIZA 3

računski del

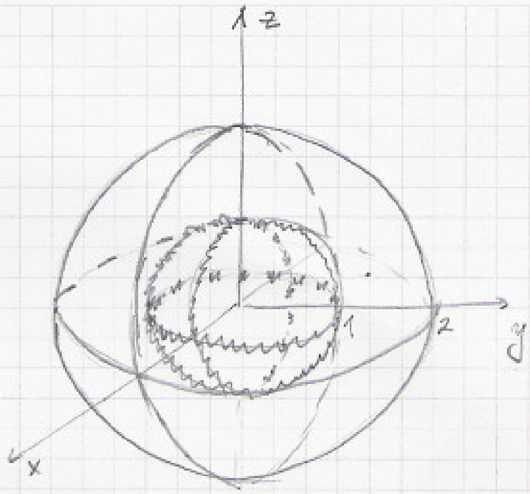
15.4.2005

Točkovanje: 20+20+35+25=100

1. Narišite skico telesa

$$G = \{(x, y, z) \in \mathbb{R}^3;\ 1 \le x^2 + y^2 + z^2 \le 4\}$$

in izračunajte njegovo maso, če je gostota sorazmerna kvadratu oddaljenosti od izhodišča.



UVEDEMO SFERIČNE KOORD.

$$x = r\cos\varphi\cos\psi$$
$$y = r\sin\varphi\cos\psi$$
$$z = r\sin\psi$$
$$J = r^2\cos\psi$$
$$\rho = C(x^2+y^2+z^2) = Cr^2$$

$$m = \iiint_G \rho(x,y,z)\,dx\,dy\,dz = \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\psi \int_1^2 \underbrace{Cr^2}_{\rho}\ \underbrace{r^2\cos\psi}_{J}\, dr$$

$$= \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\psi\, d\psi \int_1^2 r^4\, dr = 2\pi \left[\sin\psi\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{r^5}{5}\right]_1^2$$

$$= 2\pi(1-(-1))\ \frac{32-1}{5} = \boxed{\frac{124\pi}{5}}$$

2. Pokažite, da je krivulja $\vec{r}(\varphi) = (\cos\varphi, 2+\sin\varphi, 2\sin\varphi+1)$, $\varphi \in [0, 2\pi]$, ravninska. Poiščite točki na krivulji, v katerih je fleksija največja.

$$\left.\begin{aligned} \dot{\vec{r}}(\varphi) &= (-\sin\varphi, \cos\varphi, 2\cos\varphi) \\ \ddot{\vec{r}}(\varphi) &= (-\cos\varphi, -\sin\varphi, -2\sin\varphi) \\ \dddot{\vec{r}}(\varphi) &= (\sin\varphi, -\cos\varphi, -2\cos\varphi) = -\dot{\vec{r}}(\varphi) \end{aligned}\right\} \Rightarrow [\dot{\vec{r}}, \ddot{\vec{r}}, \dddot{\vec{r}}] = 0 \Rightarrow \tau = 0$$

FLEKSIJA: $\quad \varkappa(\varphi) = \dfrac{\|\dot{\vec{r}}(\varphi) \times \ddot{\vec{r}}(\varphi)\|}{\|\dot{\vec{r}}(\varphi)\|^3}$

$$\dot{\vec{r}}(\varphi) \times \ddot{\vec{r}}(\varphi) = \left|\begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin\varphi & \cos\varphi & 2\cos\varphi \\ -\cos\varphi & -\sin\varphi & -2\sin\varphi \end{matrix}\right| \begin{matrix} \vec{i} & \vec{j} \\ -\sin\varphi & \cos\varphi \\ -\cos\varphi & -\sin\varphi \end{matrix}$$

$$= (-2\cos\varphi\sin\varphi + 2\cos\varphi\sin\varphi, -2\cos^2\varphi - 2\sin^2\varphi, \sin^2\varphi + \cos^2\varphi)$$

$$= (0, -2, 1)$$

$$\Rightarrow \|\dot{\vec{r}}(\varphi) \times \ddot{\vec{r}}(\varphi)\| = \sqrt{0^2 + (-2)^2 + 1^2} = \sqrt{0+4+1} = \sqrt{5}$$

$$\|\dot{\vec{r}}(\varphi)\| = \sqrt{(-\sin\varphi)^2 + \cos^2\varphi + (2\cos\varphi)^2} = \sqrt{1 + 4\cos^2\varphi}$$

TOREJ $\quad \varkappa(\varphi) = \dfrac{\sqrt{5}}{(1+4\cos^2\varphi)^{3/2}} \quad \Longrightarrow$

$\Rightarrow$ FLEKSIJA BO NAJVEČJA, KO BO $\cos\varphi = 0$

$\Rightarrow \varphi_1 = \frac{\pi}{2}$ IN $\varphi_2 = \frac{3\pi}{2}$

$\Rightarrow T_1(\cos\frac{\pi}{2}, 2+\sin\frac{\pi}{2}, 2\sin\frac{\pi}{2}+1) = T_1(0,3,3)$

$T_2(\cos\frac{3\pi}{2}, 2+\sin\frac{3}{2}, 2\sin\frac{3}{2}+1) = T_2(0,1,-1)$

FLEKSIJA JE NAJVEČJA V TOČKAH $T_1(0,3,3)$ IN $T_2(0,1,-1)$

3. Skicirajte krivuljo $\vec{K}$, ki je presek ploskev $x^2+y^2=z$ in $z=4$ in je orientirana v smeri urinega kazalca, če jo gledamo iz T(0,0,10). Izračunajte

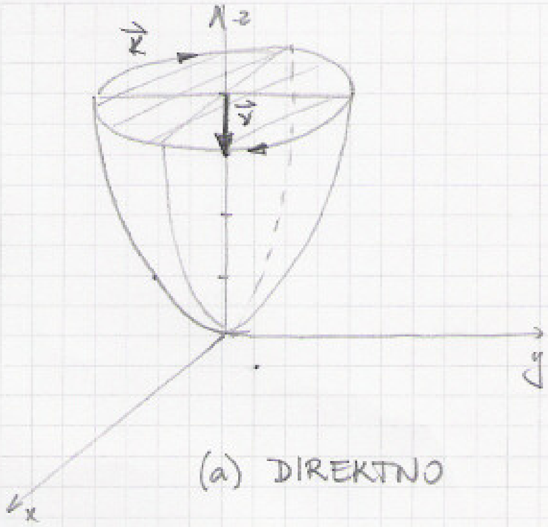
$$\oint_{\vec{K}} xz\,dy + y\,dz$$

(a) direktno,

(b) z uporabo Stokesovega izreka.

$\vec{F}(x,y,z) = (0, xz, y)$

$$\left.\begin{aligned} x^2+y^2 &= z \\ z &= 4 \end{aligned}\right\} \quad x^2+y^2=4$$



$\Rightarrow$ KRIVULJA $\vec{K}$ LEŽI V RAVNINI $z=4$, SREDIŠČE IMA V $(0,0,4)$ IN POLMER 2

(a) DIREKTNO

PARAMETRIZACIJA $\vec{K}$

$-\vec{K}: (2\cos\varphi, 2\sin\varphi, 4) \quad \varphi\in[0,2\pi]$

$$\oint_{\vec{K}} \vec{F}\,d\vec{r} = -\oint_{-\vec{K}} \vec{F}\,d\vec{r} = -\int_0^{2\pi} (0, 2\cos\varphi\cdot 4, 2\sin\varphi)(-2\sin\varphi, 2\cos\varphi, 0)\,d\varphi$$

$$= -16\int_0^{2\pi} \cos^2\varphi\,d\varphi = -\frac{16\cdot 4}{2} B\left(\tfrac{1}{2},\tfrac{3}{2}\right) = -32\frac{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})}{\Gamma(2)} = \frac{-32\sqrt{\pi}\sqrt{\pi}}{2}$$

$$= -16\pi$$

(b)

$$\operatorname{rot}\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & xz & y \end{vmatrix} \begin{matrix} i & j \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 0 & xz \end{matrix} = (1-x, 0, z)$$

$$\iint_{\Phi} \operatorname{rot}\vec{F}\,d\vec{S} = \iint_D (1-x,0,z)(0,0,-1)\,dx\,dy = -\iint_D z\,dx\,dy$$

D (KROG S POLMEROM 2 IN SREDIŠČEM V (0,0))

$$= -4\iint_D dx\,dy = -4\underbrace{(\pi\cdot 2^2)}_{\to P(D)\ \text{PLOŠČINA}} = \underline{\underline{-16\pi}}$$

4. Rešite začetni problem

$$y' - y = 1 - x \qquad y(0) = 3$$

IMAMO LINEARNO DIFERENCIALNO ENAČBO 1. REDA

NAJPREJ REŠIMO PRIREJENO HOMOGENO ENAČBO:

$$y' - y = 0$$

$$\frac{dy}{dx} = y \quad \Big| : y \qquad y=0 \text{ JE TUDI REŠITEV DE}$$

$$\frac{dy}{y} = dx$$

$$\int \frac{dy}{y} = \int dx$$

$$\ln|y| = x + \ln C \qquad C>0$$

$$\cdot \; e^{\ln|y|} = e^{x+\ln C}$$

$$|y| = Ce^x$$

$$\left.\begin{array}{l} y = \pm Ce^x \\ y = 0 \text{ JE TUDI REŠITEV} \end{array}\right\} \quad y_H = De^x \quad , D \in \mathbb{R}$$

METODA VARIACIJA KONSTANTE

$$y_P = D(x)\, e^x$$

$$y' = D' e^x + De^x$$

VSTAVIMO V D.E.

$$y' - y = 1 - x$$

$$D'e^x + De^x - De^x = 1 - x$$

$$D' = (1-x)e^{-x}$$

$$\Rightarrow D = \int (1-x)e^{-x}\,dx = (1-x)(-e^{-x}) - \int (-e^{-x})(-dx)$$

$$1 - x = u \qquad e^{-x}dx = dv$$

$$-dx = du \qquad -e^{-x} = v$$

$$= (x-1)e^{-x} + e^{-x} = xe^{-x}$$

$$\Rightarrow \quad y_P = D(x)\, e^x = (xe^{-x})e^x = x$$

$$\Rightarrow \quad y = y_H + y_P = De^x + x \quad , D \in \mathbb{R}$$

UPOŠTEVAMO ZAČETNI POGOJ:

$$\left.\begin{array}{l} y(0) = De^0 + 0 = D \\ y(0) = 3 \end{array}\right\} \Rightarrow D = 3 \quad \Rightarrow \quad \underline{\underline{y = 3e^x + x}}$$