PRIIMEK	IME	VPISNA ŠTEVILKA	SMER

NALOGA	TOČKE
1.	
2.	
3.	
4.	
SKUPAJ	

MATEMATIČNA ANALIZA 3

računski del

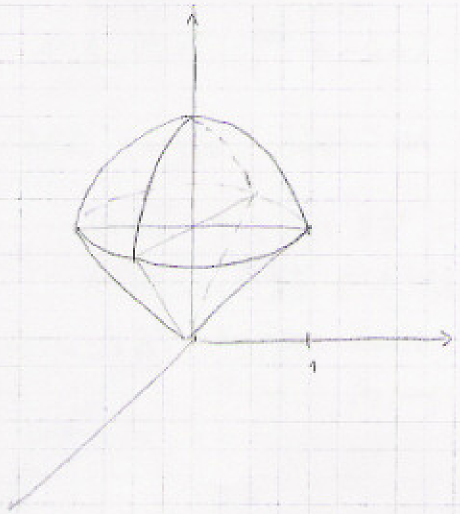
15.6.2005

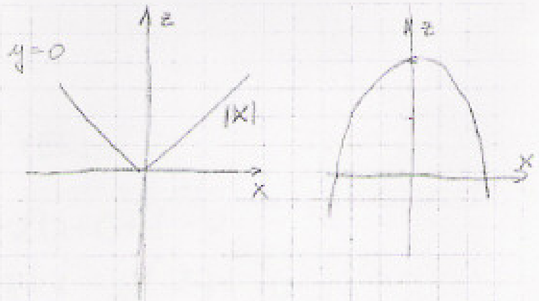
Točkovanje: 25+20+30+25=100

1. Narišite skico telesa

$$G = \{(x,y,z) \in \mathbb{R}^3;\ \sqrt{x^2+y^2} \le z \le 2 - x^2 - y^2\}$$

in izračunajte njegovo težišče.





$x^2+y^2=u^2$: $u = 2-u^2$

$u^2+u-2=0$

$(u+2)(u-1)=0 \Rightarrow u_1=1$ ✓, $u_2=-2$ //

$$m = \iiint_G dx\,dy\,dz = \int_0^{2\pi} d\varphi \int_0^1 r\,dr \int_r^{2-r^2} dz = 2\pi \int_0^1 r(2-r^2-r)\,dr$$

$x = r\cos\varphi$
$y = r\sin\varphi$
$z = z$
$J = r$

$$= 2\pi \left[\frac{2r^2}{2} - \frac{r^4}{4} - \frac{r^3}{3}\right]_0^1 = 2\pi\left[1 - \frac{1}{4} - \frac{1}{3}\right] = \frac{2\pi \cdot 5}{12} = \frac{5\pi}{6}$$

$x_T = y_T = 0$, KER JE TELO ROTACIJSKO

$$z_T = \frac{1}{m}\iiint_G z\,dz = \frac{6}{5\pi}\int_0^{2\pi} d\varphi \int_0^1 r\,dr \int_r^{2-r^2} z\,dz = \frac{6\cdot 2\pi}{5\pi}\int_0^1 r\left[\frac{z^2}{2}\right]_r^{2-r^2} dr$$

$$= \frac{6\cdot 2}{5\cdot 2}\int_0^1 r(4-4r^2+r^4-r^2)\,dr = \frac{6}{5}\int_0^1 (4r-5r^3+r^5)\,dr = \frac{6}{5}\left[\frac{4r^2}{2} - \frac{5r^4}{4} + \frac{r^6}{6}\right]_0^1$$

$$= \frac{6}{5}\left[2 - \frac{5}{4} + \frac{1}{6}\right] = \frac{6}{5}\frac{(24-15+2)}{12} = \frac{11}{10}$$

$T(0, 0, \frac{11}{10})$

2. Poiščite tisti tangentni ravnini na ploskev

$$x^2+4x+2y^2+z^2-2z+1=0,$$

ki sta vzporedni ravnini $\Sigma: x+2y+z=7$.

$$\text{grad}\,F = (2x+4,\, 4y,\, 2z-2) \Rightarrow \vec{n} = (x+2,\, 2y,\, z-1)$$

$$\vec{n}_\Sigma = (1,2,1)$$

1. POT:

$$\vec{n} \parallel \vec{n}_\Sigma \Rightarrow \vec{n} = \lambda \vec{n}_\Sigma \Rightarrow (x+2, 2y, z-1) = \lambda(1,2,1)$$

$$x+2 = \lambda \Rightarrow x = \lambda - 2$$

$$2y = 2\lambda \Rightarrow y = \lambda$$

$$z-1 = \lambda \Rightarrow z = \lambda + 1$$

KER (x,y,z) LEŽI NA PLOSKVI, VELJA

$$(\lambda-2)^2 + 4(\lambda-2) + 2\lambda^2 + (\lambda+1)^2 - 2(\lambda+1) + 1 = 0$$

$$\lambda^2 - 4\lambda + 4 + 4\lambda - 8 + 2\lambda^2 + \lambda^2 + 2\lambda + 1 - 2\lambda - 2 + 1 = 0$$

$$4\lambda^2 = 4 \Rightarrow \lambda_1 = 1 \Rightarrow T_1(-1,1,2) \Rightarrow \Sigma_1: x+2y+z = -1+2+2 = 3$$

$$\lambda_2 = -1 \Rightarrow T_2(-3,-1,0) \Rightarrow \Sigma_2: x+2y+z = -3-2 = -5$$

2. POT

$$\vec{n} \parallel \vec{n}_\Sigma \Rightarrow \vec{n} \times \vec{n}_\Sigma = \vec{0}$$

$$\vec{n}\times\vec{n}_\Sigma = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x+2 & 2y & z-1 \\ 1 & 2 & 1 \end{vmatrix}\begin{matrix} \vec{i} & \vec{j} \\ x+2 & 2y \\ 1 & 2 \end{matrix} = (2y-2(z-1),\, z-1-(x+2),\, 2(x+2)-2y) =$$

$$= (2y-2z+2,\, z-x-3,\, 2x-2y+4) = \vec{0} \Rightarrow \begin{aligned} y &= z-1 \\ x &= z-3 \\ x &= y-2 \ ✓ \end{aligned}$$

KER (x,y,z) LEŽI NA PLOSKVI, VELJA

$$(z-3)^2 + 4(z-3) + 2(z-1)^2 + z^2 - 2z + 1 = 0$$

$$z^2 - 6z + 9 + 4z - 12 + 2z^2 - 4z + 2 + z^2 - 2z + 1 = 0$$

$$4z^2 - 8z = 0 \Rightarrow z_1 = 0 \Rightarrow T_1(-3,-1,0) \Rightarrow \Sigma_1: x+2y+z = -5$$

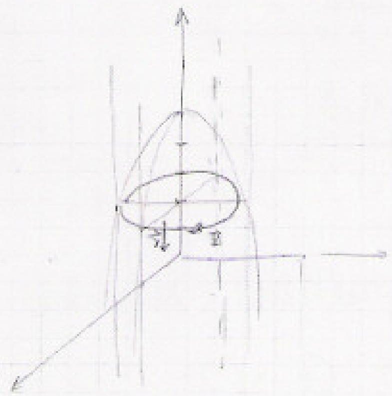
$$z_2 = 2 \Rightarrow T_2(-1,1,2) \Rightarrow \Sigma_2: x+2y+z = 3$$

3. Skicirajte krivuljo $\vec{K}$, ki je presek ploskev $x^2 + y^2 = 3$ in $z = 5 - x^2 - y^2$ in je orientirana v smeri urinega kazalca, če jo gledamo iz T(0,0,10). Izračunajte

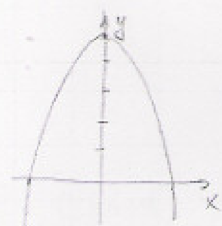
$$\oint_{\vec{K}} x^2 y\, dx + y\, dz$$

(a) direktno,

(b) z uporabo Stokesovega izreka.



$y=0: \quad z = 5 - x^2$



$\vec{F} = (x^2 y, 0, y)$

PRESEK: $x^2 + y^2 = 3$

$5 - x^2 - y^2 = z \Rightarrow z = 5 - 3 = 2$

(a) DIREKTNO: $\vec{K}: \vec{r}(t) = (\sqrt{3}\cos t, \sqrt{3}\sin t, 2)$

$\dot{\vec{r}}(t) = (-\sqrt{3}\sin t, \sqrt{3}\cos t, 0)$

$$\oint_{\vec{K}} \vec{F}\, d\vec{r} = -\int_0^{2\pi} \left((\sqrt{3}\cos t)^2 \sqrt{3}\sin t, 0, \sqrt{3}\sin t\right)(-\sqrt{3}\sin t, \sqrt{3}\cos t, 0)\, dt$$

PARAMETRIZACIJA IMA NASPROTNO ORIENTACIJO

$$= +\int_0^{2\pi} 9\cos^2 t \sin^2 t\, dt = \frac{9 \cdot 4}{2} B\left(\frac{3}{2}, \frac{3}{2}\right) = \frac{18 \cdot \frac{1}{2}\sqrt{\pi}\ \frac{1}{2}\sqrt{\pi}}{2 \cdot 1} = \frac{9\pi}{4}$$

(b) STOKES:

$$\operatorname{rot}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & 0 & y \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x^2 y & 0 \end{matrix} = (1, 0, -x^2)$$

$$\oint_{\vec{K}} \vec{F}\, d\vec{r} = \iint_{\vec{\Phi}} \operatorname{rot}\vec{F}\, d\vec{S} = -\int_0^{2\pi} d\varphi \int_0^{\sqrt{3}} (1, 0, -r^2\cos^2\varphi)(0, 0, r)\, dr$$

$\vec{f}(r, \varphi) = (r\cos\varphi, r\sin\varphi, 2) \quad \varphi \in [0, 2\pi]$

$\vec{f}_r = (\cos\varphi, \sin\varphi, 0) \quad r \in [0, \sqrt{3}]$

$\vec{f}_\varphi = (-r\sin\varphi, r\cos\varphi, 0)$

$\vec{f}_r \times \vec{f}_\varphi = (0, 0, r)$

ZA $\vec{\Phi}$ SMO VZELI KAR KROG, LAHKO PA BI VZELI TUDI PARABOLOID, ...

$$= \int_0^{2\pi} \cos^2\varphi\, d\varphi \int_0^{\sqrt{3}} r^3\, dr$$

$$= \frac{4}{2} B\left(\frac{1}{2}, \frac{3}{2}\right) \left[\frac{r^4}{4}\right]_0^{\sqrt{3}}$$

$$= \frac{2\,\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})}{\Gamma(2)} \cdot \frac{9}{4}$$

$$= \frac{2\sqrt{\pi} \cdot \sqrt{\pi} \cdot 9}{2 \cdot 4} = \frac{9\pi}{4}$$

4. Rešite začetni problem

$$y' + \frac{y}{1-x} + x^2 = 0, \qquad y(2) = -2$$

y_H: $\quad y' + \frac{y}{1-x} = 0$

$$\frac{dy}{y} = \frac{dx}{x-1}$$

$$\ln|y| = \ln|x-1| + \ln C$$

$$y = C(x-1)$$

VARIACIJA KONSTANTE: $\quad y' = C'(x-1) + C$

$$C'(x-1) + C + \frac{C(x-1)}{1-x} + x^2 = 0$$

$$C' = -\frac{x^2}{x-1} \quad = \int\left(-x-1-\frac{1}{x-1}\right)dx = -\frac{x^2}{2} - x - \ln|x-1|$$

$$x^2 : (x-1) = x+1$$
$$\underline{-x^2+x}$$
$$x$$
$$\underline{-x+1}$$
$$1$$

$$y = y_H + y_P$$
$$= C(x-1) + (x-1)\left(-\frac{x^2}{2} - x - \ln|x-1|\right)$$

$$y(2) = C + (-2-2) = -2 \Rightarrow C = 2$$

$$\underline{y = (x-1)\left(2 - \frac{x^2}{2} - x - \ln|x-1|\right)}$$