| PRIIMEK | IME | VPISNA ŠTEVILKA | SMER |
|---|---|---|---|
| | | | |

| NALOGA | TOČKE |
|---|---|
| 1. | |
| 2. | |
| 3. | |
| 4. | |
| SKUPAJ | |

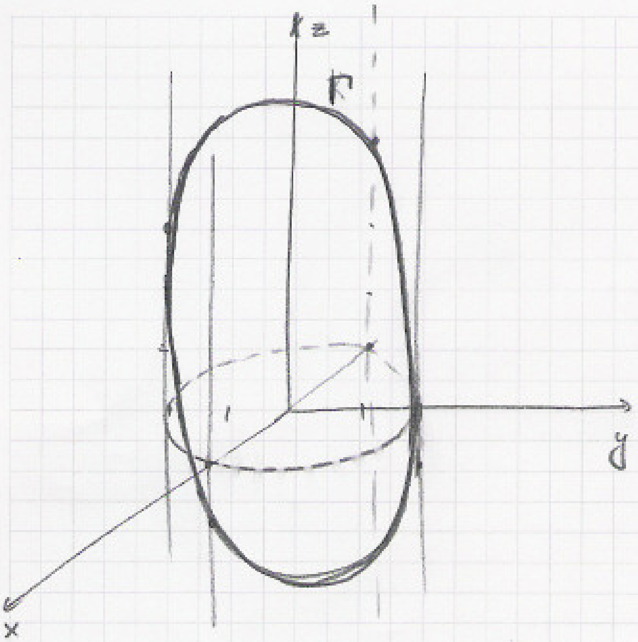
# MATEMATIČNA ANALIZA 3

računski del

3.2.2005

**Točkovanje:** 20+25+30+25=100

**1.** Skicirajte krivuljo $\Gamma = \{(x,y,z) \in \mathbb{R}^3;\ 4 = x^2 + y^2,\ x+y+z=1\}$ in jo parametrizirajte. Izračunajte fleksijo v točki $T(0,2,-1)$.




| $x$ | $y$ | $z = 1-x-y$ |
|---|---|---|
| 2 | 0 | -1 |
| 0 | 2 | -1 |
| -2 | 0 | 3 |
| 0 | -2 | 3 |

$\Gamma:\ \vec{r}(\varphi) = (2\cos\varphi,\ 2\sin\varphi,\ 1-2\cos\varphi-2\sin\varphi)$

$\varphi \in [0, 2\pi]$

$\dot{\vec{r}}(\varphi) = (-2\sin\varphi,\ 2\cos\varphi,\ 2\sin\varphi - 2\cos\varphi)$

$= 2(-\sin\varphi,\ \cos\varphi,\ \sin\varphi - \cos\varphi)$

$\ddot{\vec{r}}(\varphi) = 2(-\cos\varphi,\ -\sin\varphi,\ \cos\varphi + \sin\varphi)$

$(0,2,-1) = (2\cos\varphi,\ 2\sin\varphi,\ 1-2\cos\varphi-2\sin\varphi)$

$\Rightarrow \cos\varphi = 0,\ \sin\varphi = 1 \Rightarrow \varphi = \frac{\pi}{2}$

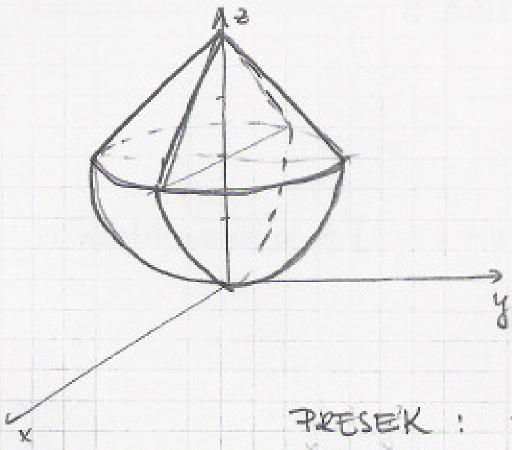
$$\left.\begin{aligned} \dot{\vec{r}}\left(\tfrac{\pi}{2}\right) &= 2(-1,0,1) \\ \ddot{\vec{r}}\left(\tfrac{\pi}{2}\right) &= 2(0,-1,1) \end{aligned}\right\} \quad \dot{\vec{r}}\left(\tfrac{\pi}{2}\right) \times \ddot{\vec{r}}\left(\tfrac{\pi}{2}\right) = 4\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{vmatrix} = 4(1,1,1)$$

$$\kappa\left(\tfrac{\pi}{2}\right) = \frac{\left\|\dot{\vec{r}}\left(\tfrac{\pi}{2}\right) \times \ddot{\vec{r}}\left(\tfrac{\pi}{2}\right)\right\|}{\left\|\dot{\vec{r}}\left(\tfrac{\pi}{2}\right)\right\|^3} = \frac{4\sqrt{1+1+1}}{\left(2\sqrt{1+0+1}\right)^3} = \frac{4\sqrt{3}}{8(\sqrt{2})^3} = \frac{\sqrt{3}}{4\sqrt{2}} = \boxed{\frac{\sqrt{6}}{8}}$$

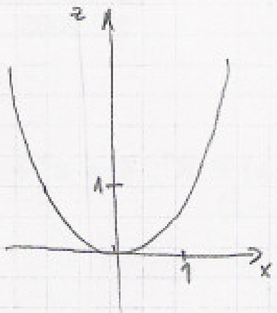
**2.** Narišite skico telesa

$$G = \{(x,y,z) \in \mathbb{R}^3;\ x^2+y^2 \le z \le 2-\sqrt{x^2+y^2}\}$$

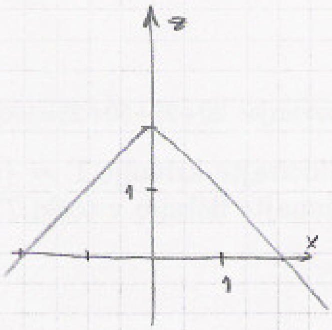
in izračunajte njegovo težišče.




$z = x^2 + y^2$




$z = 2 - \sqrt{x^2+y^2}$




PRESEK : $x^2+y^2 = 2-\sqrt{x^2+y^2}$

ČE OZNAČIMO $t = \sqrt{x^2+y^2} \geq 0$,

DOBIMO $t^2 = 2 - t \implies t^2 + t - 2 = 0$

$(t+2)(t-1) = 0$

$t_1 = -2$

$t_2 = 1 \Rightarrow x^2+y^2 = 1$ in $z = 1$

KER JE G ROTACIJSKO TELO Z OSJO z, JE $x_T = y_T = 0$

$$z_T = \frac{1}{V(G)} \iiint_G z\,dx\,dy\,dz, \text{ KER VZAMEMO } \rho = 1$$

$$V(G) = \iiint_G dx\,dy\,dz = \int_0^{2\pi} d\varphi \int_0^1 dr \int_{r^2}^{2-r} \underbrace{r}_{J}\,dz = 2\pi \int_0^1 r[z]_{r^2}^{2-r} dr = 2\pi \int_0^1 r(2-r-r^2)\,dr$$

VPELJEMO CILINDRIČNE KOORDINATE

$x = r\cos\varphi$

$y = r\sin\varphi$

$z = z$

$J = r$

$\implies$

$x^2+y^2 \le z \le 2-\sqrt{x^2+y^2}$

$r^2 \le z \le 2-r$

$$= 2\pi \int_0^1 (2r - r^2 - r^3)\,dr = 2\pi\left[r^2 - \frac{r^3}{3} - \frac{r^4}{4}\right]_0^1 = 2\pi\left(1 - \frac{1}{3} - \frac{1}{4}\right) = \frac{2\pi(12-4-3)}{12} = \frac{5\pi}{6}$$

$$z_T = \frac{1}{\frac{5\pi}{6}} \iiint_G z\,dx\,dy\,dz = \frac{6}{5\pi}\int_0^{2\pi} d\varphi \int_0^1 dr \int_{r^2}^{2-r} z \underbrace{r}_{J}\,dz = \frac{6 \cdot 2\pi}{5\pi}\int_0^1 r\left[\frac{z^2}{2}\right]_{r^2}^{2-r} dr$$

$$= \frac{6}{5}\int_0^1 r\left[(2-r)^2 - r^4\right]dr = \frac{6}{5}\int_0^1 r\left[4 - 4r + r^2 - r^4\right]dr = \frac{6}{5}\int_0^1 \left[4r - 4r^2 + r^3 - r^5\right]dr$$

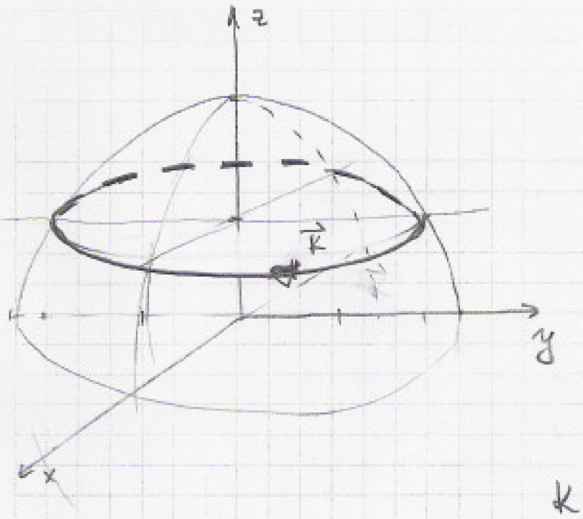
$$= \frac{6}{5}\left[\frac{4r^2}{2} - \frac{4r^3}{3} + \frac{r^4}{4} - \frac{r^6}{6}\right]_0^1 = \frac{6}{5}\left[2 - \frac{4}{3} + \frac{1}{4} - \frac{1}{6}\right] = \frac{6\,[24-16+3-2]}{5 \cdot 12} = \boxed{\frac{9}{10}}$$

**3.** Skicirajte krivuljo $\vec{K}$, ki je presek ploskev $x^2+y^2+z^2=5$ in $z=1$ in je orientirana v smeri urinega kazalca, če jo gledamo iz T(0,0,10). Izračunajte

$$\oint_{\vec{K}} y\,dx + z\,dy + xy\,dz$$

(a) direktno,
(b) z uporabo Stokesovega izreka.

$\vec{U} = (P,Q,R) = (y, z, xy)$




$x^2+y^2+z^2=5$ JE SFERA S SREDIŠČEM $(0,0,0)$ IN POLMEROM $\sqrt{5}$

PRESEK SFERE IN RAVNINE:

$$\left.\begin{matrix} x^2+y^2+z^2=5 \\ z=1 \end{matrix}\right\}$$

$$x^2+y^2=4 \Rightarrow$$

$$K = \{(x,y,z)\in\mathbb{R}^3;\ x^2+y^2=4,\ z=1\}$$

(a) PARAMETRIZACIJA $\vec{K}$: $\vec{r}(\varphi) = (2\cos\varphi, 2\sin\varphi, 1)$ $\quad \varphi\in[0,2\pi]$

$\dot{\vec{r}}(\varphi) = (-2\sin\varphi, 2\cos\varphi, 0)$

NEUSKLAJENOST ORIENTACIJ

$$\oint_{\vec{K}} y\,dx + z\,dy + xy\,dz = -\int_0^{2\pi} (2\sin\varphi, 1, 2\cos\varphi\, 2\sin\varphi)(-2\sin\varphi, 2\cos\varphi, 0)\,d\varphi$$

$$= -\int_0^{2\pi}(-4\sin^2\varphi + 2\cos\varphi)\,d\varphi = 4\int_0^{2\pi}\sin^2\varphi\,d\varphi - \underbrace{2\int_0^{2\pi}\sin\varphi\,d\varphi}_{0} = \frac{4\cdot 4}{2}B\left(\frac{3}{2},\frac{1}{2}\right)$$

$$= \frac{8\,\frac{1}{2}\sqrt{\pi}\sqrt{\pi}}{1} = \boxed{4\pi}$$

(b) $$\oint_{\vec{K}} \vec{U}\,d\vec{r} = \iint_{\vec{P}} \operatorname{rot}\vec{U}\,d\vec{S} = \iint_{\Delta}(x-1,-y,-1)(0,0,-1)\,dx\,dy$$

ZA $\vec{P}$ VZAMEMO KAR KROG: $P = \{(x,y,z)\in\mathbb{R}^3;\ x^2+y^2\le 4, z=1\}$

$$\operatorname{rot}\vec{U} = \left|\begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & xy \end{matrix}\right| \begin{matrix} \vec{i} & \vec{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ y & z \end{matrix} = (x-1, -y, -1)$$

$$\Delta = \{(x,y)\in\mathbb{R}^2;\ x^2+y^2\le 4\}$$

$$= \iint_{\Delta} dx\,dy = pl(\Delta) = \pi\cdot 2^2 = \boxed{4\pi}$$

**4.** Rešite začetni problem

$$y' - y = x \qquad y(0) = 2$$

$y' - y = x$ JE LINEARNA DIF. EN. 1. REDA

NAJPREJ REŠIMO PRIREJENO HOMOGENO L.D.E:

$$y' - y = 0$$

$$\frac{dy}{dx} = y \quad \Big| : y \qquad y = 0 \text{ JE TUDI REŠITEV}$$

$$\frac{dy}{y} = dx$$

$$\int \frac{dy}{y} = \int dx$$

$$\ln|y| = x + \ln C \quad , C > 0$$

$$|y| = Ce^x$$

$$y_H = De^x \quad , D \in \mathbb{R}$$

UPORABIMO METODO VARIACIJE KONSTANTE

$$y = D(x)e^x \Rightarrow y' = D'(x)e^x + D(x)e^x$$

VSTAVIMO V L.D.E $y' - y = x$

$$D'(x)e^x + \cancel{D(x)e^x} - \cancel{D(x)e^x} = x$$

$$D'(x) = xe^{-x} \Rightarrow$$

$$\Rightarrow D(x) = \int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = (-x-1)e^{-x}$$

$$x = u \qquad e^{-x}dx = dv$$

$$dx = du \qquad -e^{-x} = v$$

$$\Rightarrow y_P = D(x)e^x = (-x-1)e^{-x} \cdot e^x = -x-1$$

$$\Rightarrow y = y_H + y_P = De^x - x - 1$$

UPORABIMO ZAČETNI POGOJ $y(0) = 2$:

$$y(0) = De^0 - 0 - 1 = D - 1 = 2 \Rightarrow D = 3$$

$$\Rightarrow \boxed{y = 3e^x - x - 1} \quad \text{( REŠITEV ZAČETNEGA PROBLEMA)}$$