PRIIMEK	IME	VPISNA ŠTEVILKA	SMER

NALOGA	TOČKE
1.	
2.	
3.	
4.	
SKUPAJ	

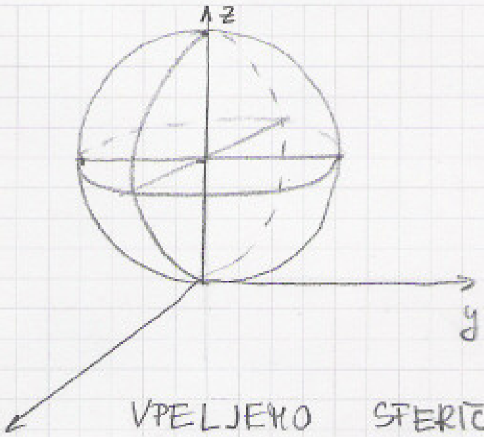
MATEMATIČNA ANALIZA 3

računski del

26.1.2005

Točkovanje: 25+20+30+25=100

1. Narišite skico telesa
$$G = \{(x,y,z) \in \mathbb{R}^3;\ x^2+y^2+z^2 \le 2z\}$$
in izračunajte njegovo maso, če je gostota sorazmerna oddaljenosti od ravnine $z=0$.



$$x^2+y^2+z^2 \le 2z$$
$$x^2+y^2+z^2-2z+1 \le 1$$
$$x^2+y^2+(z-1)^2 \le 1$$

SFERA S SREDIŠČEM V $S(0,0,1)$
IN POLMEROM $r=1$

VPELJEMO SFERIČNE KOORDINATE

$$x = r\cos\varphi\cos\psi$$
$$y = r\sin\varphi\cos\psi$$
$$z = r\sin\psi$$
$$J = r^2\cos\psi$$

$$\underbrace{x^2+y^2+z^2}_{r^2} \le 2z$$
$$r^2 \le 2r\sin\psi$$
$$r \le 2\sin\psi$$

$$\varphi \in [0,2\pi],\ \psi \in [0,\tfrac{\pi}{2}],\ r \in [0, 2\sin\psi]$$

$\rho(x,y,z) = k|z| = kz$, KER G LEŽI NAD RAVNINO $z=0$

$$m = \iiint_G \rho(x,y,z)\,dx\,dy\,dz = \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} d\psi \int_0^{2\sin\psi} \underbrace{kr\sin\psi}_{\rho}\ \underbrace{r^2\cos\psi}_{J}\,dr$$

$$= k\cdot 2\pi \int_0^{\frac{\pi}{2}} \sin\psi\cos\psi \left[\frac{r^4}{4}\right]_0^{2\sin\psi} =$$

$$= \frac{k\cdot 2\pi\cdot \cancel{16}\,4}{\cancel{4}} \int_0^{\frac{\pi}{2}} \cos\psi \sin^5\psi\,d\psi = \frac{8k\pi}{2}\,B(3,1) = \frac{4k\pi\,\Gamma(3)\Gamma(1)}{\Gamma(4)}$$

$$= \frac{4\cdot k\pi\cdot\cancel{2}\cdot 1}{3\cdot\cancel{2}\cdot 1} = \boxed{\frac{4k\pi}{3}}$$

2. Poiščite tiste tangentne ravnine ploskve

$$x^2 + y^2 + 4z^2 + 2y - 8 = 0,$$

ki so vzporedne ravnini $\Pi : x + 2y + 4z = 5$.

$\vec{n}_\Pi = (1, 2, 4)$

$\vec{n} \parallel \operatorname{grad} \vec{F} = (2x, 2y+2, 8z) = 2(x, y+1, 4z)$

$\vec{n} \parallel \vec{n}_\Pi \Rightarrow (x, y+1, 4z) = \lambda(1, 2, 4) \Rightarrow \left.\begin{aligned} x &= \lambda \\ y+1 &= 2\lambda \\ 4z &= 4\lambda \end{aligned}\right\} \Rightarrow$

$x = \lambda,\ y = 2\lambda - 1,\ z = \lambda$

KER $(x, y, z) \subset P$, VELJA

$$x^2 + y^2 + 4z^2 + 2y - 8 = 0$$
$$\lambda^2 + (2\lambda-1)^2 + 4\lambda^2 + 2(2\lambda - 1) - 8 = 0$$
$$\lambda^2 + 4\lambda^2 - \cancel{4\lambda} + 1 + 4\lambda^2 + \cancel{4\lambda} - 2 - 8 = 0$$
$$9\lambda^2 - 9$$
$$\lambda^2 = 1 \Rightarrow \lambda = \pm 1 \Rightarrow T_1(1, 1, 1)$$
$$T_2(-1, -3, -1)$$

Σ_1: $x + 2y + 4z = (1, 2, 4)\vec{r}_{T_1} = (1, 2, 4)(1, 1, 1) = 7$

$\underline{x + 2y + 4z = 7}$

Σ_2: $x + 2y + 4z = (1, 2, 4)\vec{r}_{T_2} = (1, 2, 4)(-1, -3, -1) = -1 - 6 - 4 = -11$

$\underline{x + 2y + 4z = -11}$

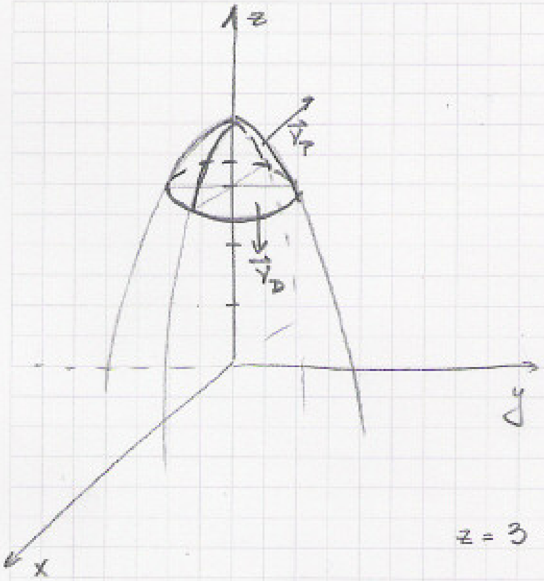
3. Izračunajte pretok vektorskega polja $\vec{U}\,(x,y,z) = (y,x,3)$ skozi zgornjo stran paraboloida

$$P = \{(x,y,z) \in \mathbb{R}^3; z = 4 - x^2 - y^2, 3 \le z \le 4\}.$$

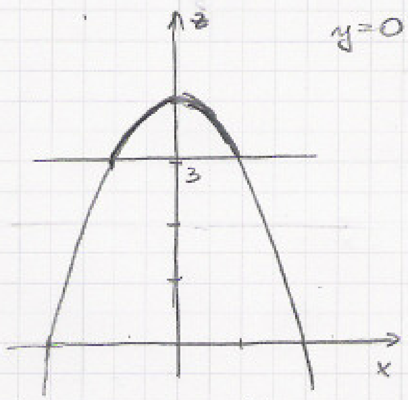
(a) direktno

(b) z uporabo Gaussovega izreka

Skicirajte ploskev $\vec{P}$.



$z = 4 - x^2 - y^2 \; ; \quad 3 \le z \le 4$



$z = 3: \; x^2 + y^2 = 1 \quad ; \quad z = 4: \; x^2 + y^2 = 0$

(a) $\vec{f}(r,\varphi) = (r\cos\varphi, r\sin\varphi, 4 - r^2) \qquad \varphi \in [0, 2\pi], \; r \in [0,1]$

$$\left.\begin{aligned} \vec{f}_r &= (\cos\varphi, \sin\varphi, -2r) \\ \vec{f}_\varphi &= (-r\sin\varphi, r\cos\varphi, 0) \end{aligned}\right\} \vec{f}_r \times \vec{f}_\varphi = (2r^2\cos\varphi, 2r^2\sin\varphi, r) \Longrightarrow$$

$\vec{f}_r \times \vec{f}_\varphi$ IN $\vec{\nu}_1$ KAŽETA V ISTO SMER

$$\iint_{\vec{P}} \vec{U}\,d\vec{S} = +\int_0^{2\pi} d\varphi \int_0^1 (r\sin\varphi, r\cos\varphi, 3)(2r^2\cos\varphi, 2r^2\sin\varphi, r)\,dr$$

$$= \int_0^{2\pi} d\varphi \int_0^1 (2r^3\sin\varphi\cos\varphi + 2r^3\sin\varphi\cos\varphi + 3r)\,dr$$

$$= 4\underbrace{\int_0^{2\pi} \sin\varphi\cos\varphi\,d\varphi}_{0} \int_0^1 r^3\,dr + 3\int_0^{2\pi} d\varphi \int_0^1 r\,dr = 0 + 3\cdot 2\pi \left[\frac{r^2}{2}\right]_0^1 = \underline{\underline{3\pi}}$$

ALI

EKSPLICITNO

$$\left.\begin{aligned} z &= 4 - x^2 - y^2 \\ p &= z_x = -2x \\ q &= z_y = -2y \end{aligned}\right\} \quad (-p, -q, 1) = (2x, 2y, 1)$$

$$\iint_{\vec{P}} \vec{U}\,\vec{dS} = + \iint_{\Delta} (y, x, 3)\overbrace{(2x, 2y, 1)}^{(-p,-q,1)}\,dx\,dy$$

$$\Delta = \{(x,y) \in \mathbb{R}^2 ;\ x^2+y^2 \leq 1\}$$

$$= \iint_{\Delta} (2xy + 2xy + 3)\,dx\,dy \quad = \int_0^{2\pi} d\varphi \int_0^1 (4r^2\cos\varphi\sin\varphi + 3)\,r\,dr$$

POLARNE KOORDINATE

$$x = r\cos\varphi$$
$$y = r\sin\varphi$$
$$J = r$$

$$= \underbrace{4\int_0^{2\pi} \cos\varphi\sin\varphi\,d\varphi}_{0} \int_0^1 r^3\,dr + 3\int_0^{2\pi} d\varphi \int_0^1 r\,dr$$

$$= 3 \cdot 2\pi \left[\frac{r^2}{2}\right]_0^1 = \underline{\underline{3\pi}}$$

(B) $\vec{P}$ DOPOLNIMO z $\vec{D} = \{(x,y,3) \in \mathbb{R}^3 ;\ x^2+y^2 \leq 1\}$, $\vec{\nu}_D = (0,0,-1)$

$$\iint_{\vec{P}+\vec{D}} \vec{U}\,\vec{dS} = \iiint_G \operatorname{div}\vec{U}\,dV \qquad \operatorname{div}\vec{U} = 0+0+0 = 0$$

$$\Rightarrow \iint_{\vec{P}} \vec{U}\,\vec{dS} = \iiint_G \underbrace{\operatorname{div}\vec{U}}_{0}\,dV - \iint_{\vec{D}} \vec{U}\,\vec{dS}$$

$$= 0 - \iint_{\vec{D}} \vec{U}\,\vec{dS}$$

$$= -\iint_{\Delta} (y, x, 3)(0,0,-1)\,dx\,dy$$

$$\Delta = \{(x,y) \in \mathbb{R}^2 ;\ x^2+y^2 \leq 1\}$$

$$= -\iint_{\Delta} (-3)\,dx\,dy = 3\iint_{\Delta} dx\,dy$$

$$= 3\,pl(\Delta) = 3 \cdot \pi \cdot 1^2 = \underline{\underline{3\pi}}$$

4. Poiščite diferencialno enačbo družine

$$Cx^2 + y^2 = 1$$

in med ortogonalnimi trajektorijami poiščite tisto, ki gre skozi točko $T(1,1)$.

① POIŠČIMO DE DRUŽINE $Cx^2+y^2=1 \Rightarrow C = \frac{1-y^2}{x^2}$

ODVAJAMO : $2Cx + 2yy' = 0 \Rightarrow y' = -\frac{Cx}{y} = -\frac{(1-y^2)x}{x^2y} = \frac{y^2-1}{xy}$

② $y' \rightsquigarrow -\frac{1}{y'}$

$y' = -\frac{xy}{y^2-1} = \frac{xy}{1-y^2}$ (DE ORTOGONALNIH TRAJEKTORIJ)

③ REŠIMO $y' = \frac{xy}{1-y^2} \quad \Big| : \frac{1-y^2}{y}$ $y=0$ JE TUDI REŠITEV

$$\frac{(1-y^2)dy}{y} = x\,dx$$

$$\int \frac{1}{y}\,dy - \int y\,dy = \int x\,dx$$

$$\ln|y| - \frac{y^2}{2} = \frac{x^2}{2} + C \quad , C \in \mathbb{R}$$

$$\ln|y| = \frac{x^2+y^2}{2} + C$$

$$|y| = De^{\frac{x^2+y^2}{2}} \quad D>0$$

$$y = \pm De^{\frac{x^2+y^2}{2}}$$

$$\underline{y = Ee^{\frac{x^2+y^2}{2}}}$$, KER JE $y=0$ TUDI REŠITEV.

④ ORTOGONALNA TRAJEKTORIJA SKOZI $T(1,1)$

$$1 = Ee^{\frac{1^2+1^2}{2}} = Ee \Rightarrow E = e^{-1}$$

$$\Rightarrow \boxed{y = e^{\frac{x^2+y^2-2}{2}}}$$