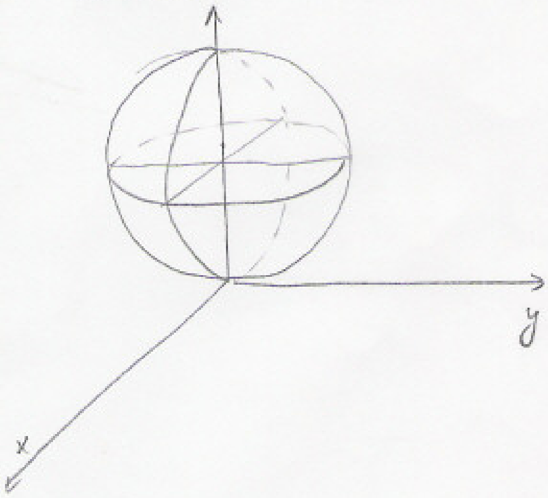
DN. 10

IZRAČUNAJ MASO KROGLE $G = \{(x,y,z);\, x^2+y^2+z^2 \leq z\}$,

ČE JE GOSTOTA SORAZMERNA ODDALJENOSTI OD IZHODIŠČA



$$x^2+y^2+z^2-z+\frac{1}{4}=\frac{1}{4}$$

$$x^2+y^2+\left(z-\frac{1}{2}\right)^2=\left(\frac{1}{2}\right)^2$$

UVEDEMO SFERIČNE KOORDINATE

$x = r\cos\psi\cos\varphi$

$y = r\cos\psi\sin\varphi$

$z = r\sin\psi$

$J = r^2\cos\psi$

$\rho = kr$

LAHKO TUDI:

$x = r\sin\psi\cos\varphi$

$y = r\sin\psi\sin\varphi$

$z = r\cos\psi$

$J = r^2\sin\psi$

MEJE: $\varphi \in [0, 2\pi]$

$\psi \in [0, \frac{\pi}{2}]$

$\underbrace{x^2+y^2+z^2}_{} = z$ $\qquad \sqrt{x^2+y^2+z^2} = r$

$r^2 = r\sin\psi$ (KER $r \neq 0$)

$r = \sin\psi$

$$I = \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} d\psi \int_0^{\sin\psi} \underset{\rho}{kr}\, \underbrace{r^2\cos\psi}_{\text{Jac. det.}}\, dr = \cancel{2}\pi k \int_0^{\frac{\pi}{2}} \cos\psi \left[\frac{r^4}{\cancel{4}2}\right]_0^{\sin\psi} d\psi$$

$$= \frac{\pi k}{2}\int_0^{\frac{\pi}{2}} \sin^4\psi\cos\psi\, d\psi = k\frac{\pi}{2}\cdot\frac{1}{2}B\left(\frac{5}{2},1\right) = \frac{k\pi}{4}\,\frac{\Gamma(\frac{5}{2})\Gamma(1)}{\Gamma(\frac{7}{2})} =$$

$$= \frac{k\pi}{4}\,\frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}} = \boxed{\frac{k\pi}{10}}$$