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Domača naloga

Zamenjaj vrstni red in izračunaj integral:

$$I = \int_0^{\pi/4} dy \int_{\operatorname{tg} y}^{\frac{2}{\sqrt{\pi}}\sqrt{\pi - y}} \frac{4}{1+x^2}\,dx$$

Zamenjava vrstnega reda:

$$\int_0^{\pi/4} dy \int_{\operatorname{tg} y}^{\frac{2}{\sqrt{\pi}}\sqrt{\pi - y}} f(x,y)\,dx = *$$

(I)

$$x = \operatorname{tg} y$$

$$y = \operatorname{arctg} x$$

(II)

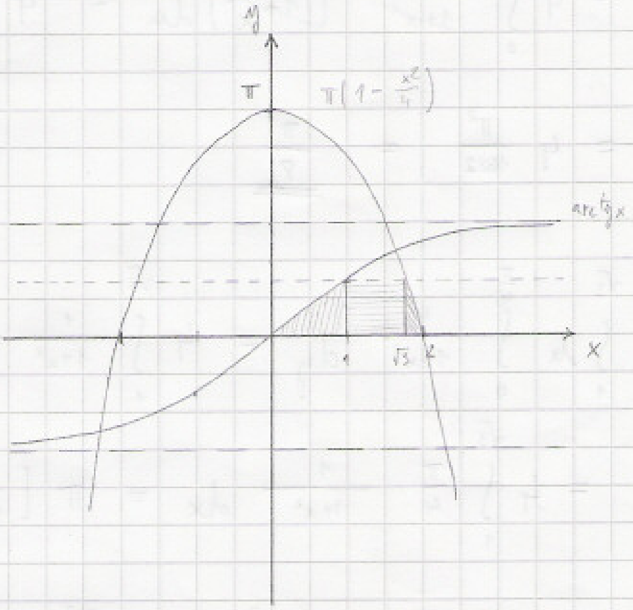
$$x = \frac{2}{\sqrt{\pi}}\sqrt{\pi - y}$$

$$\frac{x\sqrt{\pi}}{2} = \sqrt{\pi - y}$$

$$\frac{x^2\sqrt{\pi}}{4} = \pi - y$$

$$y = \pi - \frac{x^2\pi}{4}$$

$$y = \pi\left(1 - \frac{x^2}{4}\right) = \frac{\pi(4-x^2)}{4}$$



$$\operatorname{tg} y = \frac{\pi}{4} \quad \rightarrow \quad y = 1$$

$$\frac{2}{\sqrt{\pi}}\sqrt{\pi - \frac{\pi}{4}} = \frac{2}{\sqrt{\pi}}\sqrt{\frac{3\pi}{4}} = \sqrt{3}$$

$$* = \int_0^1 dx \int_0^{\operatorname{arctg} x} f(x,y)\,dy + \int_1^{\sqrt{3}} dx \int_0^{\frac{\pi}{4}} f(x,y)\,dy + \int_{\sqrt{3}}^2 dx \int_0^{\pi(1-\frac{x^2}{4})} f(x,y)\,dy$$

Izračun integrala:

I. $$\int_0^1 dx \int_0^{\operatorname{arctg} x} \frac{4}{1+x^2}\, dy = 4\int_0^1 \frac{1}{1+x^2}\left[\, y \,\right]_0^{\operatorname{arctg} x} dx =$$

$$= 4\int_0^1 \frac{\operatorname{arctg} x}{1+x^2}\, dx =$$

$$\operatorname{arctg} x = u \qquad x = 1 \rightarrow u = \frac{\pi}{4}$$

$$\frac{1}{1+x^2}\, dx = du \qquad x = 0 \rightarrow u = 0$$

$$= 4\int_0^{\frac{\pi}{4}} \frac{u}{\cancel{1+x^2}} \cancel{(1+x^2)}\, du = 4\left[\frac{u^2}{2}\right]_0^{\frac{\pi}{4}} =$$

$$= 4\,\frac{\pi^2}{16 \cdot 2} = \underline{\underline{\frac{\pi^2}{8}}}$$

II. $$\int_1^{\sqrt{3}} dx \int_0^{\frac{\pi}{4}} \frac{4}{1+x^2}\, dy = 4\int_1^{\sqrt{3}} \frac{1}{1+x^2}\left[\, y \,\right]_0^{\pi/4} dx =$$

$$= \cancel{4}\int_1^{\sqrt{3}} \frac{\pi}{\cancel{4}} \cdot \frac{1}{1+x^2}\, dx = \pi\left[\operatorname{arctg} x\right]_1^{\sqrt{3}} =$$

$$= \pi\left[\frac{\pi}{3} - \frac{\pi}{4}\right] = \frac{\pi^2}{3} - \frac{\pi^2}{4} = \underline{\underline{\frac{\pi^2}{12}}}$$

$$\text{III.}\quad \int_{\sqrt{3}}^{2} dx \int_{0}^{\frac{\pi(4-x^2)}{4}} \frac{4}{1+x^2}\, dy = 4\int_{\sqrt{3}}^{2} \frac{1}{1+x^2} \left[y \right]_0^{\frac{\pi(4-x^2)}{4}} dx =$$

$$= \cancel{4}\int_{\sqrt{3}}^{2} \frac{\pi(4-x^2)}{\cancel{4}} \cdot \frac{1}{1+x^2}\, dx = \pi \int_{\sqrt{3}}^{2} \frac{4-x^2}{1+x^2}\, dx =$$

$$\frac{4-x^2}{1+x^2} = \frac{A}{1} + \frac{B}{1+x^2} =$$

$$= \frac{(1+x^2)A + B}{1+x^2} = \frac{A + Ax^2 + B}{1+x^2}$$

$A + B = 4$ $\qquad$ $A = -1$, $B = 5$

$$= -1 + \frac{5}{1+x^2}$$

$$= \pi\left(\int_{\sqrt{3}}^{2} \frac{5}{1+x^2}\, dx - \int_{\sqrt{3}}^{2} dx \right) = 5\pi \left[\operatorname{arctg} \right]_{\sqrt{3}}^{2} - \pi \left[x \right]_{\sqrt{3}}^{2} =$$

$$= 5\pi \left(\operatorname{arctg} 2 - \frac{\pi}{3} \right) - \pi\left(2 - \sqrt{3}\right) =$$

$$= 5\pi \operatorname{arctg} 2 - \frac{5\pi^2}{3} - \pi\left(2-\sqrt{3}\right)$$

Seštejemo vse tri rešitve:

$$I = \frac{\pi^2}{8} + \frac{\pi^2}{12} + 5\pi \operatorname{arctg} 2 - \frac{5\pi^2}{3} - \pi\left(2-\sqrt{3}\right)$$

$$= 5\pi \operatorname{arctg} 2 - \pi\left(2-\sqrt{3}\right) + \frac{3\pi^2 + 2\pi^2 - 40\pi^2}{24}$$

$$= 5\pi \operatorname{arctg} 2 - \pi\left(2-\sqrt{3}\right) - \frac{35\pi^2}{24}$$