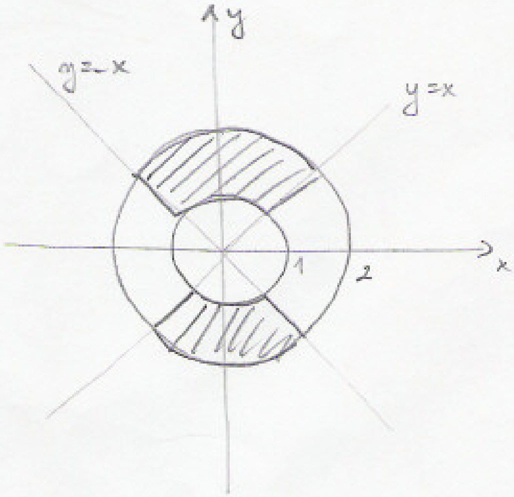
DN #4

4

IZRAČUNAJ $I = \iint\limits_D \frac{dx\,dy}{(x^2+y^2)(1+\sqrt[3]{x^2+y^2})}$, KJER JE

$$D = \{(x,y);\ x^2-y^2 \le 0;\ 1 \le x^2+y^2 \le 4\}$$



$$x^2 - y^2 \le 0$$
$$(x-y)(x+y) \le 0$$

① $x - y \ge 0 \Rightarrow x \ge y$
$x + y \le 0 \Rightarrow x \le -y$

② $x - y \le 0 \Rightarrow x \le y$
$x + y \ge 0 \Rightarrow x \ge -y$

UVEDEMO POLARNE KOORDINATE

$x = r\cos\varphi$
$y = r\sin\varphi$
$J = r$

KER JE $f(x,y) = \frac{1}{(x^2+y^2)(1+\sqrt[3]{x^2+y^2})} = f(x,-y) = f(-x,y) = f(-x,-y)$
JE DOVOLJ ČE RAČUNAMO SAMO PO DELU V I. KVADRANTU
IN GA POMNOŽIMO S 4

$$I = 4\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\varphi \int_1^2 \frac{\overbrace{r\,dr}^{=\frac{3}{2}t^2dt}}{r^2(1+\sqrt[3]{r^2})} = 4\varphi\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_1^{\sqrt[3]{4}} \frac{\frac{3}{2}t^2\,dt}{t^2(1+t)} =$$

$\sqrt[3]{r^2} = t$

$r^2 = t^3$

$2r\,dr = 3t^2dt$

$$= \frac{\cancel{4}\cdot\pi\cdot 3}{\cancel{4}\cdot 2}\int_1^{\sqrt[3]{4}} \frac{dt}{t(1+t)}$$

$$\frac{1}{t(1+t)} = \frac{A}{t} + \frac{B}{1+t} = \frac{A+At+Bt}{t(1+t)} \Rightarrow \begin{array}{l} A=1 \\ A+B=0 \Rightarrow \\ \Rightarrow B=-1 \end{array}$$

$$= \frac{3\pi}{2}\left(\int_1^{\sqrt[3]{4}} \frac{1}{t}\,dt - \int_1^{\sqrt[3]{4}} \frac{1}{1+t}\,dt\right) = \frac{3\pi}{2}\left(\ln|t| - \ln|1+t|\right)\Big|_1^{\sqrt[3]{4}}$$

$$= \frac{3\pi}{2}\ln\frac{t}{1+t}\Big|_1^{\sqrt[3]{4}} = \frac{3\pi}{2}\left(\ln\frac{\sqrt[3]{4}}{1+\sqrt[3]{4}} - \ln\frac{1}{2}\right) = \frac{3\pi}{2}\ln\frac{2\sqrt[3]{4}}{1+\sqrt[3]{4}} = \boxed{\frac{\pi}{2}\ln\frac{32}{(1+\sqrt[3]{4})^3}}$$