Etale categories, Ehresmann quantales and restriction semigroups

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Applications of operator algebras: order, disorder and symmetry
ICMS workshop Edinburgh
June 29, 2017
Stone duality for Boolean inverse semigroups
The duality between finite sets and finite Boolean algebras:

- \( X \) — finite set, \( \mathcal{P}(X) \) — finite Boolean algebra. 
  \( X \leftrightarrow \) atoms of \( \mathcal{P}(X) \).

- \( f : X \to Y \) — set map, then \( f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X) \) is a morphism of BAs.

- \( f : \mathcal{P}(X) \to \mathcal{P}(Y) \) — a morphism of BAs, then \( f^{-1} \) maps atoms of \( \mathcal{P}(Y) \) to atoms of \( \mathcal{P}(X) \).

This can be generalized to \( X \) infinite in two ways:

1. The duality between sets and complete and atomic Boolean algebras.
2. Stone duality for Boolean algebras.
"Non-commutative" example

Let $X$ be a finite set and

- $X \times X$ be the pair groupoid of $X$.
- $\mathcal{P}(X \times X)$ — the powerset quantale of $X \times X$.

Either of these structures allows to recover the other one (powerset, atoms). Furthermore:

- $\mathcal{I}(X)$ be the symmetric inverse monoid on $X$ (bisections of $X \times X$).
- $M_{|X|}(\mathbb{C})$ the full matrix algebra of size $|X|$.

Generalizations to $X$ infinite:

1. Duality between infinite symmetric inverse semigroups and "discrete groupoids" (mediated by respective quantales).
2. Stone duality for Boolean inverse semigroups (mediated by respective "finitary" quantales).
Boolean inverse semigroups

Let $S$ be an inverse semigroup. For $a \in S$ we set $d(a) = a^{-1}a$ and $r(a) = aa^{-1}$.

$a, b \in S$ are called compatible if $ad(b) = bd(a)$ and $r(b)a = r(a)b$, denoted $a \sim b$.

$S$ is called a Boolean inverse semigroup if

- $E(S)$ admits a structure of a Boolean algebra.
- If $a \sim b$ then their join $a \lor b$ exists in $S$.

Boolean inverse semigroups are distributive: for any $a \in S$ and compatible $b, c \in S$ we have that $ab \sim ac$ and $ba \sim ca$ and

$$a(b \lor c) = ab \lor ac; \quad (b \lor c)a = ba \lor ca.$$ 

A Boolean inverse semigroup $S$ is called a a Boolean inverse $\land$-semigroup, if it is also a meet semilattice with respect to the natural partial order on $S$. 
A **Boolean groupoid** is a topological étale groupoid whose space of identities is a Boolean space.

A **(local) bisection** of a Boolean groupoid \( G \) is a compact open subset \( B \subseteq G \) such that the restriction of both domain and range maps to \( B \) are injective.

\[ G \quad \text{— Boolean groupoid} \quad \Rightarrow \quad Bi(G) \quad \text{— Boolean inverse semigroup.} \]

\[ G \quad \text{— Hausdorff Boolean groupoid} \quad \Rightarrow \quad Bi(G) \quad \text{— Boolean inverse \( \land \)-semigroup.} \]
Theorem (M. V. Lawson, 2009, 2011)

*The category of Boolean inverse $\land$-semigroups is dual to the category of Hausdorff Boolean groupoids.*

Theorem (M. V. Lawson and D. H. Lenz, 2011)

*The category of Boolean inverse semigroups is dual to the category of Boolean groupoids.*
Stone duality for Boolean restriction semigroups
Let $X$ be a finite set

- $\rho$ – a reflexive and transitive binary relation on $X$.
- A subsemigroup of $I(X)$ consisting of those $\varphi$ for which $i \in \text{dom}(\varphi)$ implies that $i \rho \varphi(i)$.
- $\mathcal{P}(\rho)$ the powerset quantale of $\rho$.
- (Non-selfadjoint) subalgebra of $M_{|X|}(\mathbb{C})$ generated by the matrix units $e_{i\rho(i)}$ (the digraph algebra).

An observation
Either of these structures allows to recover the other ones.

Generalization to $X$ infinite:
Stone duality for Boolean restriction semigroups.
Restriction semigroups are non-regular generalizations of inverse semigroups, they are equipped with two unary operations $a \mapsto a^*$ and $a \mapsto a^+$ satisfying certain axioms which mimic taking $d(a)$ and $r(a)$ in an inverse semigroup.

**Theorem (GK and M. V. Lawson, 2014)**

*The category of Boolean restriction semigroups is dual to the category of Boolean étale categories.*
Research direction: Connect non-selfadjoint subalgebras of groupoid $C^*$-algebras (e.g. AF-algebras or graph algebras) with respective étale subcategories of étale groupoids, and also with underlying restriction subsemigroups of inverse semigroups.

For (certain) subalgebras of AF-algebras Power (1990) showed that they are classified by ”topological binary relations” (= principal étale categories) which are contained in ”topological equivalence relations” (=principal étale groupoids).
Frames, locales and Stone-type dualities
(P. Johnstone "Stone spaces")
Pointless topology studies lattices with properties similar to the properties of lattices of open sets of topological spaces.

Pointless topology studies lattices $L$ which are

- **sup-lattices**: for any $x_i \in L$, $i \in I$, their join $\bigvee x_i$ exists in $L$.
- **infinitely distributive**: for any $x_i \in L$, $i \in I$, and $y \in L$

\[
y \land (\bigvee_{i \in I} x_i) = \bigvee_{i \in I} (y \land x_i).
\]

Such lattices are called **frames**.

- A **frame morphism** $\varphi : F_1 \to F_2$ is required to preserve finite meets and arbitrary joins.
The category of **locales** is defined to be the opposite category to the category of frames. Locales are ‘pointless topological spaces’.

**Notation**

If $L$ is a locale then $O(L)$ is the **frame of opens** of $L$.

A locale morphism $\varphi : L_1 \to L_2$ is defined as the frame morphism $\varphi^* : O(L_2) \to O(L_1)$.
If $L$ is a locale then points of $L$ are defined as frame morphisms $L \rightarrow \{0, 1\}$. Topology on $\text{pt}(L)$ is the subspace topology inherited from the product space $\{0, 1\}^L$.
This gives rise to the spectrum functor

$$\text{pt} : \text{Loc} \rightarrow \text{Top}.$$ 

Assigning to a topological space its frame of opens leads to the functor

$$\Omega : \text{Top} \rightarrow \text{Loc}.$$ 

**Theorem**

*The functor $\text{pt}$ is the right adjoint to the functor $\Omega$.***

Is this adjunction an equivalence?
No!
Spatial frames and sober spaces

- A space $X$ is **sober** if $\text{pt}(\Omega(X)) \simeq X$.
- A locale $F$ is **spatial** if $\Omega(\text{pt}(F)) \simeq F$.

**Theorem**

The above adjunction restricts to an equivalence between the categories of spatial locales and sober spaces.

**Example of a non-sober space:**

$\{1, 2\}$ with indiscrete topology.

**Example of non-spatial frame:**

A complete non-atomic Boolean algebra, for example the Boolean algebra of Lebesgue measurable subsets of $\mathbb{R}$ modulo the ideal of sets of measure 0.
Coherent frames and distributive lattices

A space $X$ is called **spectral** if it is sober and compact-open sets form a basis of the topology closed under finite intersections. A frame is called **coherent** if it is isomorphic to a frame of ideals of a distributive lattice.

**Theorem**

The following categories are pairwise equivalent:

- The category of distributive lattices
- The category of coherent frames
- The opposite of the category of spectral spaces

A **Boolean space** is a Hausdorff spectral space.

**Theorem**

The following categories are equivalent:

- The category of Boolean algebras
- The category of Boolean spaces
Ehresmann quantal frames and quantal localic categories
A quantale \((Q, \leq, \cdot)\) is a sup-lattice \((Q, \leq)\) equipped with a binary multiplication operation \(\cdot\) such that multiplication distributes over arbitrary suprema:

\[
a(\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (ab_i) \text{ and } (\bigvee_{i \in I} b_i)a = \bigvee_{i \in I} (b_ia).
\]

A quantale is **unital** if there is a multiplicative unit \(e\) and **involutive**, if there is an involution \(*\) on \(Q\) which is a sup-lattice endomorphism.

A **quantal frame** is a quantale which is also a frame.
A unital quantale $Q$ with unit $e$ is called an \textbf{Ehresmann quantale} if there are two maps $\lambda, \rho : Q \to Q$ such that

(E1) both $\lambda$ and $\rho$ are sup-lattice endomorphisms;

(E2) if $a \leq e$ then $\lambda(a) = \rho(a) = a$;

(E3) $a = \rho(a)a$ and $a = a\lambda(a)$ for all $a \in Q$;

(E4) $\lambda(ab) = \lambda(\lambda(a)b), \rho(ab) = \rho(a\rho(b))$ for all $a, b \in Q$.

Under multiplication, they are \textbf{Ehresmann semigroups}, introduced and first studied by Mark Lawson in 1991.

\textbf{Another notation:} $\lambda(a) = a^*, \rho(a) = a^+$.

An \textbf{Ehresmann quantal frame} is an Ehresmann quantale that is also a frame.
Example

- $X$ – non-empty set
- $A \subseteq X \times X$ – a transitive and reflexive relation
- $\mathcal{P}(A)$ – the powerset of $A$
- $e$ – the identity relation

For $a \in \mathcal{P}(A)$ we define

$$a^* = \{(x, x) \in X \times X : \exists y \in X \text{ such that } (y, x) \in a\} \in e^\perp,$$

$$a^+ = \{(y, y) \in X \times X : \exists x \in X \text{ such that } (y, x) \in a\} \in e^\perp.$$

This can be generalized to the Ehresmann quantal frame of opens of a topological (or localic) category.
A localic category is an internal category in the category of locales. That is, we are given the data

\[ C = (C_1, C_0, u, d, r, m), \text{ or } C = (C_1, C_0), \] for short, where \( C_1 \) is a locale, called the locale of arrows, and \( C_0 \) is a locale, called the locale of objects, together with four locale maps

\[ u : C_0 \to C_1, \quad d, r : C_1 \to C_0, \quad m : C_1 \times C_0 C_1 \to C_1, \]

called unit, domain, codomain, and multiplication, respectively. \( C_1 \times_{C_0} C_1 \) is the object of composable pairs defined by the pullback diagram

\[
\begin{array}{ccc}
C_1 \times_{C_0} C_1 & \xrightarrow{\pi_2} & C_1 \\
\downarrow{\pi_1} & & \downarrow{r} \\
C_1 & \xrightarrow{d} & C_0
\end{array}
\]
Localic categories

The four maps $u, d, r, m$ are subject to conditions that express the usual axioms of a category:

1. $du = ru = id$.
2. $m(u \times id) = \pi_2$ and $m(id \times u) = \pi_1$.
3. $r\pi_1 = rm$ and $d\pi_2 = dm$.
4. $m(id \times m) = m(m \times id)$. 
Ehresman quantal frames vs étale localic categories: dictionary

### Commutative setting

<table>
<thead>
<tr>
<th>Frame</th>
<th>Locale</th>
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<tr>
<td>$O(L)$</td>
<td>$L$</td>
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### Non-commutative setting

<table>
<thead>
<tr>
<th>Quantal frame $Q$</th>
<th>Étale localic category $C = (C_1, C_0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q = O(C_1)$</td>
<td>locale $C_1$</td>
</tr>
<tr>
<td>$e^\downarrow = O(C_0)$</td>
<td>locale $C_0$</td>
</tr>
<tr>
<td>quantale multiplication of $Q$</td>
<td>category multiplication of $C$</td>
</tr>
<tr>
<td>$\cdot, +: Q \to e^\downarrow$</td>
<td>domain and range maps $d$ and $r$ of $C$</td>
</tr>
<tr>
<td>Ehresmann multiplicative: properties of $\cdot, \ast$ and $+$</td>
<td>quantal: properties of $d$ and $r$</td>
</tr>
<tr>
<td>restriction: properties of $\ast$ and $+$ partial isometries generate $Q$</td>
<td>étale: properties of $d$ and $r$</td>
</tr>
</tbody>
</table>
Maps between locales

A locale map $f : L \to M$ is called **semiopen** if the defining frame map $f^* : O(M) \to O(L)$ preserves arbitrary meets. Then the left adjoint

$$f_! : O(L) \to O(M)$$

to $f^*$ is called the **direct image map** of $f$.

$f$ is called **open** if the **Frobenius condition** holds:

$$f_!(a \land f^*(b)) = f_!(a) \land b$$

for all $a \in O(L)$ and $b \in O(M)$.

**Example:** if $f : X \to Y$ is an open continuous map between topological spaces then it is open as a locale map.
An an Ehresmann quantal frame $Q$ is *multiplicative* if the right adjoint $m^*$ of the multiplication map

$$Q \otimes_{e} Q \to Q$$

preserves arbitrary joins and thus the multiplication map is a direct image map of a locale map.

An étale localic category $C = (C_1, C_0, u, d, r, m)$ is *quantal* if the maps $u, d, r$ are open and $m$ is semiopen (that is, $m_!$ exists and $m$ can be ‘globalized’).

**Correspondence Theorem (GK and M. V. Lawson, 2014)**

There is a bijective correspondence between multiplicative Ehresmann quantal frames and quantal localic categories.
A morphism $\varphi : Q_1 \to Q_2$ between Ehresmann quantal frames is a quantale map that is also a map of Ehresmann monoids (preserves both $\ast$ and $\oplus$).

We consider the following four types of morphisms between Ehresmann quantal quantal frames:

- **type 1**: morphisms;
- **type 2**: proper morphisms (unital morphism);
- **type 3**: $\land$-morphisms (preserves non-empty finite meets);
- **type 4**: proper $\land$-morphisms (preserves all finite meets).

In the multiplicative case, morphisms between respective quantal localic categories are defined as the above morphisms but going in the opposite direction. Thus the Correspondence Theorem becomes a categorical duality.
Restriction quantal frames, complete restriction monoids and étale localic categories
Partial isometries

- $Q$ – an Ehresmann quantal frame
- $a \in Q$
- $a$ is a partial isometry if $b \leq a$ implies that $b = af = ga$ for some $f, g \leq e$
- Notation: $\mathcal{PI}(Q)$

Example

$X$ a non-empty set, $A \subseteq X \times X$ a transitive and reflexive relation. The partial isometries of the Ehresmann quantal frame $\mathcal{P}(A)$ are precisely partial bijections.
Étale correspondence theorem

A localic category $C = (C_1, C_0)$ is étale if $u, m$ are open and $d, r$ are local homeomorphisms.

An Ehresmann quantal frame $Q$ is a restriction quantal frame if every element is a join of partial isometries and partial isometries are closed under multiplication.

**Theorem (GK and M. V. Lawson, 2014)**

The category of restriction quantal frames is dually equivalent to the category of étale localic categories.

**Remark:** This extends and is inspired by the correspondence between inverse quantal frames and étale localic groupoids due to Pedro Resende.
Complete restriction monoids

Restriction quantal frames

Complete restriction monoids

Étale localic categories
Restriction semigroups form a subclass of Ehresmann semigroups. They satisfy:

\[ a^* b = b(ab)^*, \quad ba^+ = (ba)^+ b \]

for all \( a, b \in S \).

- \( S \) is complete if \( E \) is a complete lattice and joins of compatible families of elements exist in \( S \).

**Theorem**

The category of complete restriction monoids is equivalent to the category of restriction quantal frames.

This extends an equivalence between pseudogroups and inverse quantal frames established by Pedro Resende.
Down to topological dualities
The adjunction

**Theorem**
There is an adjunction between:
- the category of étale localic categories and
- the category of étale topological categories.

This adjunction is given by the spectrum and open set functors and extends the classical adjunction between locales and topological spaces.

**Corollary**
There is a dual adjunction between:
- the category restriction quantal frames and
- the category of étale topological categories.

This adjunction extends the classical dual adjunction between frames and topological spaces.
Let $C = (C_1, C_0)$ be an étale localic category. Then the locale $C_1$ is spatial iff the locale $C_0$ is spatial. If these hold $C$ is called spatial.

Let $C = (C_1, C_0)$ be an étale topological category. Then the space $C_1$ is sober iff the space $C_0$ is sober. If these hold $C$ is called sober.

**Corollary**

The category of spatial étale localic categories is equivalent to the category of sober étale topological categories.
RS = restriction semigroups

<table>
<thead>
<tr>
<th>Algebraic object</th>
<th>Topological étale category $C = (C_1, C_0)$</th>
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<tbody>
<tr>
<td>Distributive RS</td>
<td>$C_0$ – spectral</td>
</tr>
<tr>
<td>Distributive $\land$ RS</td>
<td>$C_1$ (and thus also $C_0$) spectral</td>
</tr>
<tr>
<td>Boolean RS</td>
<td>$C_0$ – Boolean</td>
</tr>
<tr>
<td>Boolean $\land$ RS</td>
<td>$C_1$ (and thus also $C_0$) Boolean</td>
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</table>

**Remark.** Restriction semigroup $\rightarrow$ inverse semigroup, category $\rightarrow$ groupoid.
Another approach to the notion of a "non-commutative space"
Skew Boolean algebras (introduced by Jonathan Leech in 1989) are "non-commutative analogues" of Boolean algebras where \(\land\) and \(\lor\) are idempotent but no longer commutative. A left-handed skew Boolean algebra satisfies \(x \land y \land z = x \land z \land y\) (left normality). (Note that the operations are everywhere defined, also SBAs can be defined by identities and thus form a variety of algebras, free skew Boolean algebras - GK and J. Leech, 2016, free skew Boolean intersection algebras – GK, 2016.)

**Theorem** (GK; Cvetko-Vah and Bauer, 2012)
The category of skew Boolean algebras (skew Boolean \(\land\)-algebras) is dual to the category of étale spaces with global support (Hausdorff étale spaces with global support) over Boolean spaces.
## Skew Boolean algebras vs Boolean inverse semigroups

<table>
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<th>Band setting</th>
<th>Inverse semigroup setting</th>
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<td>Skew Boolean algebras</td>
<td>Biases (Fred Wehrung)</td>
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<tr>
<td>Boolean left normal bands</td>
<td>Boolean inverse semigroups</td>
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<tr>
<td>Étale spaces over Boolean spaces</td>
<td>Étale groupoids with Boolean spaces of identities</td>
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</thead>
<tbody>
<tr>
<td>Skew Boolean intersection algebras</td>
<td>Biases with meets</td>
</tr>
<tr>
<td>Boolean left normal bands with meets</td>
<td>Boolean inverse meet semigroups</td>
</tr>
<tr>
<td>Hausdorff Étale spaces over Boolean spaces</td>
<td>Hausdorff Étale groupoids with Boolean spaces of identities</td>
</tr>
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</table>
## Distributive skew Boolean algebras vs distributive Boolean inverse semigroups

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</tr>
<tr>
<td>Étale spaces over spectral spaces</td>
<td>Étale groupoids with spectral spaces of identities</td>
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<tbody>
<tr>
<td>Distributive skew lattices</td>
<td>?</td>
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<tr>
<td>Étale spaces over Priestley spaces</td>
<td>?</td>
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**Skew frames:** recent work of Karin Cvetko-Vah, **non-commutative topos:** recent work by Karin Cvetko-Vah, Jens Hemelaer, Lieven Le Bruyn